An Introduction to Algorithmic Symbolic Integration

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0 Introduction

Symbolic integration is a topic studied by most undergraduate students of mathematics. Students are taught a variety of heuristics and are encouraged to view finding integrals as an exercise in selecting the appropriate trick with the appropriate parameters to solve the problem at hand. Although this serves as a useful and informative introduction to the topic, it has lead to a general understanding that this is the only way to evaluate integrals symbolically; that there are no effective algorithms to perform symbolic integration. It is understandable therefore that we are often amazed that computer algebra software can easily solve integrals that look very difficult.

Example 0.1. Following the methods described in this paper, I have written code in the Magma computer algebra system to integrate fractions of polynomials. This implementation is able to solve the following easily:

$$\begin{aligned}\int \frac{9x^{11} - 12x^{10} - 47x^9 + 52x^8 + 51x^7 - 62x^6 + 21x^5 - 24x^4 - 2x^3 + 18x^2 + 16x - 16}{x^{12} - x^{11} - 7x^{10} + 7x^9 + 9x^8 - 9x^7 + 9x^6 - 9x^5 - 8x^4 + 8x^3}\,dx \\ = \log(x^3 - x^2 + x - 1) + (1+\sqrt{2})\log(x^3 + \sqrt{2}x^2 - 3x - 2(1+\sqrt{2}) + 2) \\ + (1-\sqrt{2})\log(x^3 + (1-\sqrt{2})x^2 - 3x + 2(1+\sqrt{2}) - 2) + \frac{1}{x^2}.\end{aligned} \tag{1}$$

■

For the purposes of this paper, we are only interested in algorithms for solving integrals *exactly* (which is not to say that there is never any value in numerical approximation). This has two particular implications:

1. Computing with decimals (e.g. floating point numbers) will be insufficient. Some rational numbers have no finite decimal representation (in some fixed base), so we will store rational numbers by storing two integers. Every irrational number has an infinite decimal representation in every base. Therefore we will limit ourselves to using the rational numbers and numbers which are algebraic over the rationals. In the latter case, there is a sensible way to represent these numbers in computer algebra systems and to work inside fields containing them (which we will not explore).

2. We cannot use the method of constructing and evaluating limits to compute integrals. Instead, we will build on the well-known methods of using symbols to represent functions from analysis

and finding symbolic anti-derivatives, thus using algebra to encode an inherently geometric and analytic problem. We use the terms *integral*, *symbolic antiderivative*, and *symbolic integral* interchangeably.

The goal of this paper is to provide an introduction to the field of algorithmic symbolic integration for students who may not be familiar with any abstract algebra. The main focus will be on rational integration, but with a short excursion into the wider field of transcendental integration. We will use Magma code to perform computation in several examples, with a view towards the reader being able to extend the provided code to write a complete and effective algorithm for rational integration. An implementation of rational integration in Magma, along with a partial implementation of logarithmic and exponential integration, may be found on `https://github.com/mitchellholt/magma-integration`.

This paper is organised as follows:

1. Section 1 introduces some important concepts from abstract algebra that are necessary to begin discussing symbolic integration;
2. Section 2 gives a derivation for existing methods of rational integration and states (without proof) the *Rothstein-Trager Theorem*;
3. Section 3 states and proves some fundamental theorems for transcendental integration and explores selected parts of the *Risch procedure*.

1 Algebraic Preliminaries

Before we can discuss symbolic integration, we need to introduce some necessary language and objects from abstract algebra.

1.1 Fields

Definition 1.1 (Field)**.** A *field* is a set F with binary operations called addition $(+)$ and multiplication $(\cdot)$, a zero element 0, and identity element 1 satisfying:

(F1) Both addition and multiplication are associative and commutative.

(F2) For all $a, b, c \in F$, we have $a \cdot (b + c) = a \cdot b + a \cdot c$.

(F3) For all $a \in F$, we have $a + 0 = a$.

(F4) For all $a \in F - \{0\}$, we have $a \cdot 1 = a$.

(F5) For all $a \in F$, there exists an element $(-a) \in F$ such that $a + (-a) = 0$.

(F6) For all $a \in F - \{0\}$, there exists an element $a^{-1} \in F$ such that $aa^{-1} = 1$.

(F7) $0 \neq 1$.

It is usually understood from context what the zero and identity elements and addition and multiplication operations of F are.

Fields are a class of objects we are generally very familiar with.

Example 1.1. The set of rational numbers $\mathbb{Q}$ is a field with zero element 0, identity 1, and the usual addition and multiplication operations. In particular, any non-zero $\frac{p}{q} \in \mathbb{Q}$ has $(\frac{p}{q})^{-1} = \frac{q}{p}$. ■

Example 1.2. The set of integers $\mathbb{Z}$ is *not* a field with the usual addition, multiplication, zero, and identity because (F6) fails (notice that there is no integer k with $2k = 1$). We *can* construct a field containing the integers by taking the *field of fractions* $\mathcal{F}(\mathbb{Z}) = \{\frac{n}{m} \mid n, m \in \mathbb{Z}, m \neq 0\} = \mathbb{Q}$. We extend the addition and multiplication operations from $\mathbb{Z}$ to $\mathcal{F}(\mathbb{Z})$ in the natural way and have the usual rules for equality of fractions $\frac{p}{q} = \frac{kp}{kq}$ for all $p, q, k \in \mathbb{Z}$, $q \neq 0$. ■

The definition given in *Example 1.2* is not quite the same as the formal definition for the field of fractions. However, for our purposes, it is "good enough".

Example 1.3. Similarly to *Example 1.2*, the polynomials with coefficients from $\mathbb{Q}$ (we write this as $\mathbb{Q}[x]$) is not a field. However, we can "complete" $\mathbb{Q}[x]$ to a field by taking the field of fractions $\mathbb{Q}(x) := \mathcal{F}(\mathbb{Q}[x]) = \{\frac{f}{g} \mid f, g \in \mathbb{Q}[x], g \neq 0\}$. ■

Example 1.4. $\mathbb{Q}(i) = \{a + bi \mid a, b \in \mathbb{Q}\}$ (where $i^2 = -1$) is a field. ■

Often in abstract algebra, we are interested not only in objects themselves, but in the relationships between them. This is a useful tool that we will apply when we go on to discuss transcendental integration.

Definition 1.2 (Extension Field)**.** Let F and G be fields such that $F \subseteq G$ (as sets). If the restriction of the addition and multiplication operations of G to F are equal to the addition and multiplication operations of F respectively, then we say that G is an *extension* of F and that F is a *subfield* of G.

Example 1.5. $\mathbb{Q}$ is a subfield of $\mathbb{R}$, which is a subfield of $\mathbb{C}$. ■

Example 1.6. $\mathbb{Q}(i)$ is the *smallest* subfield of $\mathbb{C}$ containing both $\mathbb{Q}$ and i. ■

Example 1.7. $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$ is the smallest subfield of $\mathbb{C}$ containing both $\mathbb{Q}$ and $\sqrt{2}$. Note that, although we can describe $\mathbb{Q}(i)$ and $\mathbb{Q}(\sqrt{2})$ similarly, they are different fields with different properties. ■

Example 1.8. Let $S \subseteq \mathbb{C}$. Then $\mathbb{Q}(S)$ is the smallest subfield of $\mathbb{C}$ containing both $\mathbb{Q}$ and S. If S is finite (and possibly empty) with $S = \{s_1, \ldots, s_n\}$, then we may write $\mathbb{Q}(S)$ as $\mathbb{Q}(s_1, \ldots, s_n)$ and call this field a *finitely generated extension* of $\mathbb{Q}$. ■

Example 1.8 is of particular importance to us. We will only work with constants coming from a finitely generated extension of $\mathbb{Q}$. The finite set generating this extension will always be one containing *algebraic numbers*.

Definition 1.3 (Algebraic)**.** Let F be a field and $g \in G$, where G is an extension of F. If there exists a polynomial $p \in F[x]$ such that $p(g) = 0$, then we say that g is *algebraic* over F. If g is not algebraic, we say that it is *transcendental*.

Example 1.9. Although the imaginary unit i is *not* a rational number, it is a root of the polynomial $x^2 + 1 \in \mathbb{Q}[x]$, so it is *algebraic* over $\mathbb{Q}$. ■

Example 1.10. Baker [1] shows that the constant $\pi \in \mathbb{C}$ is *transcendental* over $\mathbb{Q}$. ■

Example 1.11. Let $\mathbb{Q}(x, y)$ be the *field of fractions* of the multivariate polynomial ring $\mathbb{Q}[x, y]$. Observe that $\mathbb{Q}(x)$ is a subfield of $\mathbb{Q}(x, y)$, but the symbol y is *transcendental* over $\mathbb{Q}(x)$. ■

1.2 Differential Fields

Differential fields are perhaps the central object in the study of algorithmic symbolic integration.

Definition 1.4 (Differential Field)**.** A *differential field* (F, D) is a field F with a *differential map* $D : F \to F$ such that, for all $f, g \in F$:

(i) $D(f + g) = Df + Dg$; and

(ii) $D(fg) = D(f)g + fD(g)$.

We call the subfield $\{f \in F \mid Df = 0\}$ the *constant field* or *field of constants* of F.

It is a fairly straight-forward exercise to prove that the constant field is indeed a subfield of F. Consider *Proposition 1.1*, in which we construct a differential field of rational functions and show that differentiating monomials behaves in the way we would expect.

Proposition 1.1. *Equip $\mathbb{Q}(x)$ with the standard derivative map $'$; that is, $q' = 0$ for all $q \in \mathbb{Q}$ and $x' = 1$. For all $a \in \mathbb{Q}$ and $n \in \mathbb{N}$, we have $(ax^n)' = nax^{n-1}$.*

Proof. We proceed by induction. For the base case, take $n = 0$. Then $(ax^0)' = a' = 0 = 0ax^{-1}$. Now suppose that $(ax^k)' = kax^{k-1}$ for some $k \in \mathbb{N}$. Then $(ax^{k+1})' = (ax^k)' \cdot x + ax^k \cdot x' = kax^k + ax^k = (k + 1)ax^k$ as desired. $\square$

Definition 1.5 (Differential Extension)**.** Let (F, D) and (G, E) be differential fields. We say that (G, E) is a *differential extension* of (F, D) if G is an extension field of F and $E|_F = D$ (where $E|_F$ denotes the restriction of E to F).

1.3 Polynomials

We work with polynomials extensively in symbolic integration. We are particularly interested in some key properties.

Definition 1.6 (Degree)**.** Let F be a field and $f \in F[x] - \{0\}$ with $f = \sum_{i=0}^{n} f_i x^i$ for some $n \in \mathbb{N}$ and $f_0, \ldots, f_n \in F$ such that $f_n \neq 0$. The *degree* of f (denoted $\deg f$) is n, the highest power of x appearing in f. Conventions for $\deg 0$ vary, so whenever we write $\deg p$ we implicitly assume that p is non-zero.

We state *Theorem 1.1* without proof. Most algebra textbooks should have a proof of this fact.

Theorem 1.1. *Let F be a field and $f, g \in F[x] - \{0\}$. Then* $\deg(fg) = \deg f + \deg g$.

Definition 1.7 (Monic)**.** Let F be a field and $f \in F[x]$ with $f = \sum_{i=0}^{n} f_i x^i$ for some $n \in \mathbb{N}$ and $f_0, \ldots, f_n \in F$, $f_n \neq 0$. We say f is *monic* if $f_n = 1$.

Definition 1.8 (GCD)**.** Let F be a field and $f, g \in F[x]$. The *greatest common divisor* $\gcd(f,\, g)$ is the polynomial, maximum with respect to degree, amongst all monic polynomials dividing both f and g. By convention $\gcd(0,\, 0) = 0$.

Example 1.12. In $\mathbb{Q}[x]$, $\gcd(x-1, x^2+1) = x-1$ (since $x - 1 \mid x^2 + 1$). ■

Example 1.13. In $\mathbb{Q}[x]$, $\gcd(x^3 + 4x^2 - 2x,\, x^4 - x^3 - 20x^2 + 18x - 4) = x^2 + 4x - 2$. ■

Definition 1.9 (Square-free)**.** Let F be a field. We say a polynomial $f \in F[x]$ is *square-free* if there is no non-zero polynomial $p \in F[x]$ with $\deg p > 0$ and $p^2 \mid f$.

Example 1.14. Let F be a field. In $F[x]$, every polynomial with degree at most one is square-free. To see this, suppose $p \in \mathbb{Q}[x]$ with $\deg p \leq 1$. We observe that, for non-zero polynomials $a, b \in F[x]$, $a \mid b$ implies that $\deg a \leq \deg b$. Moreover, for any non-constant polynomial $f \in \mathbb{Q}[x]$, $\deg f^2 = 2 \deg f > 1$. Therefore f^2 cannot divide p, so p is square-free. ■

Example 1.15. In $\mathbb{Q}[x]$, $x^2 + 2x + 1 = (x+1)^2$ is *not* square-free. ■

Definition 1.10 (Irreducible)**.** Let F be a field. We say that a polynomial $p \in F[x]$ is irreducible if $p = fg$ implies that $\deg f = 0$ or $\deg g = 0$.

An irreducible polynomial can be thought of as being analogous to a prime number, which has the same properties in the integers. Indeed, if F is a field, then a polynomial $p \in F[x]$ can be written as a unique product of irreducibles (up to scaling by elements of F).

Example 1.16. If F is a field, then every polynomial $p \in F[x]$ with $\deg p = 1$ is irreducible. To demonstrate this, suppose $p = fg$ for some $f, g \in F[x] - \{0\}$. Then $\deg(fg) = \deg f + \deg g = 1$, so one of f, g must have degree 0. ■

2 Rational Integration

Before we can discuss rational integration, we need to define what it means to find an integral in the context of a differential field.

Definition 2.1 (Rational Integral)**.** Let $(\mathbb{Q}(x), ')$ be the rational functions under the usual derivative (with $q' = 0$ for all $q \in \mathbb{Q}$ and $x' = 1$). Let (G, D) be a differential extension of $(\mathbb{Q}(x), ')$. We say that $g \in G$ is the *integral* of $f \in \mathbb{Q}(x)$ if $g' = f$ and $G = K(x, \theta_1, \ldots, \theta_m)$, where K is the constant field of G and is a *finitely generated algebraic extension* of $\mathbb{Q}$, and each θ_i is *transcendental* over $\mathbb{Q}(x)$ with derivative $D\theta_i = \frac{u_i'}{u_i}$ for some $u_i \in \mathbb{Q}(x)$. If this is the case, we write $g = \int f$ and $\theta_i = \log u_i$.

Notice that we write $\int f$ and not $\int f\,dx$. This is because the antiderivative is determined completely by the differential field we are working in, making the dx unnecessary. The notation $g = \int f$ in *Definition 2.1* is somewhat misleading as we make no claims about the uniqueness of antiderivatives (indeed, if $g = \int f$, then $g + c = \int f$ for all $c \in K$). We nonetheless use this notation because it is the convention for symbolic integration.

Example 2.1. Consider $3x^2 + 4x + 5 \in \mathbb{Q}(x)$. Notice that $x^3 + 2x^2 + 5x - 12 \in \mathbb{Q}(x)$ has $(x^3 + 2x^2 + 5x - 12)' = 3x^2 + 4x + 5$. Therefore $\int 3x^2 + 4x + 5 = x^3 + 2x^2 + 5x - 12$. ■

Example 2.2. Consider $\frac{1}{x} \in \mathbb{Q}(x)$. Let θ be transcendental over $\mathbb{Q}(x)$ with $\theta' = \frac{x'}{x} = \frac{1}{x}$. Therefore $\int \frac{1}{x} = \theta \in \mathbb{Q}(x, \theta)$. We may write $\theta = \log u$ when $\theta' = \frac{u'}{u}$ for some $u \in \mathbb{Q}(x)$. ■

We prove some fundamental properties of integrals which we would expect to be true.

Proposition 2.1 (Properties of Integrals)**.** *Let $f, g \in \mathbb{Q}(x)$ and (G, D) be as in* Definition 2.1 *such that $\int f, \int g \in G$. The following hold:*

(i) *$\int(f + g) \in G$ and $\int(f + g) = \int f + \int g$.*

(ii) *$\int kf \in G$ and $\int kf = k \int f$ for all $k \in \mathbb{Q}$.*

Proof. For (i), observe that $D(\int f + \int g) = D(\int f) + D(\int g) = f + g$, so $\int(f + g) = \int f + \int g \in G$. For (ii), we have $D(k \int f) = kD(\int f) + Dk \int f = kf$, so $\int kf = k \int f \in G$. □

A natural place for us to begin writing an algorithm for rational integration is to examine the tools we already have.

Example 2.3. Let $p \in \mathbb{Q}(x)$ be a polynomial; that is, $p = \sum_{i=0}^{n} p_i x^i$ for some $n \in \mathbb{N}$ and $p_0, \ldots, p_n \in \mathbb{Q}$ with $p_n \neq 0$. From *Proposition 1.1* we know that $(ax^k)' = kax^{k-1}$ for all $a \in \mathbb{Q}$ and $k \in \mathbb{N}$. Therefore

$$\int ax^k = \frac{ax^{k+1}}{k+1} \in \mathbb{Q}(x), \tag{2}$$

which we can easily assert is correct by taking the derivative. It follows from *Proposition 2.1* that

$$\int p = \int \sum_{i=0}^{n} p_i x^i = \sum_{i=0}^{n} \frac{p_i x^{i+1}}{i+1} \in \mathbb{Q}(x). \tag{3}$$

■

Example 2.4. A natural observation following *Definition 2.1* is that, for all $u \in \mathbb{Q}(x)$,

$$\int \frac{u'}{u} = \log u \in \mathbb{Q}(x, \log u). \tag{4}$$

■

Our strategy for developing an algorithm for rational integration is to reduce every problem to the case of either *Example 2.3* or *Example 2.4*.

2.1 Partial Fractions

Recall that, when F is a field, each polynomial $p \in F[x]$ has a unique factorisation into irreducibles.

Example 2.5. Consider $\frac{1}{x^2-3x+2} \in \mathbb{Q}(x)$. Observe that the denominator $x^2-3x+2 = (x-1)(x-2)$. Using *Example 1.14*, we see that this is indeed the (unique) factorisation of $x^2 - 3x + 2$ into irreducibles. Consider the equation

$$\frac{1}{x^2 - 3x + 2} = \frac{A}{x-1} + \frac{B}{x-2}, \tag{5}$$

which has the unique solution $A = -B = -1$. Integrating Equation (5), we find that

$$\begin{aligned} \int \frac{1}{x^2 - 3x + 2} &= \int \frac{-1}{x-1} + \int \frac{1}{x-2} \\ &= -\log(x-1) + \log(x-2) \quad \in \mathbb{Q}(x, \log(x-1), \log(x-2)). \end{aligned} \tag{6}$$

■

This technique is called *integration by partial fractions* or simply *partial fractions*.

Example 2.6. Consider $\frac{1}{x^2-2} \in \mathbb{Q}(x)$, which has an irreducible denominator (in $\mathbb{Q}[x]$). To apply partial fractions (as in *Example 2.5*), we need to expand the constant field $\mathbb{Q}$ to include a root of $x^2 - 2$. Following *Example 1.7*, we know that $\mathbb{Q}(\sqrt{2})$ is the smallest such field. Working in $\mathbb{Q}(\sqrt{2})(x)$, we can now use partial fractions to find that

$$\int \frac{1}{x^2 - 2} = \int \frac{\frac{\sqrt{2}}{4}}{x - \sqrt{2}} - \int \frac{\frac{\sqrt{2}}{4}}{x + \sqrt{2}} = \tfrac{\sqrt{2}}{4} \log(x - \sqrt{2}) - \tfrac{\sqrt{2}}{4} \log(x + \sqrt{2}), \tag{7}$$

which lies in $\mathbb{Q}(\sqrt{2})(x,\, \log(x-\sqrt{2}),\, \log(x+\sqrt{2}))$. ■

Having seen that partial fractions allows us to compute integrals in some simple examples, the natural question to ask is if this generalises nicely. Consider *Theorem 2.1*, which we present without proof.

Theorem 2.1 (Partial Fractions)**.** *Let $f, g \in \mathbb{Q}[x] - \{0\}$. Write g be the product of distinct irreducibles $g = \prod_{i=1}^{k} p_i^{n_i}$. There are unique b, a_{ij} with $\deg a_{ij} < \deg p_i$ such that*

$$\frac{f}{g} = b + \sum_{i=1}^{k} \sum_{j=1}^{n_i} \frac{a_{ij}}{p_i^j}. \tag{8}$$

We call such a decomposition the partial fraction decomposition *of $\frac{f}{g}$.*

While *Theorem 2.1* is certainly useful, and there are efficient algorithms to compute this partial fraction decomposition, this is not quite sufficient to completely solve the problem of rational integration.

Example 2.7. Consider $\frac{1}{x^2} \in \mathbb{Q}(x)$. By *Theorem 2.1*, this is the (unique) partial fraction decomposition of $\frac{1}{x^2}$. However, there is no $u \in \mathbb{Q}(x)$ with $\frac{1}{x^2} = \frac{u'}{u}$. ■

Observe that in both *Example 2.5* and *Example 2.6*, the rational functions we wish to integrate have *square-free* denominators (see *Definition 1.9*). However, in *Example 2.7*, the denominator x^2 is *not* square free! For now, let us merely restrict ourselves to finding integrals of rational functions whose denominator (when written in lowest terms) is square-free.

Theorem 2.2 (Kronecker)**.** *Let F be a field and $p \in F[x]$ with $\deg p > 0$. There exists an extension G of F such that p has a root in G.*

Corollary 2.1. *There exists an extension K of F such that p has $\deg p$ roots in K. We call such a field K a* splitting field *of p. Moreover, every polynomial has a* minimal *splitting field (with respect to inclusion).*

Let $\frac{f}{g} \in \mathbb{Q}(x) - \{0\}$ such that g is square-free. Without loss of generality, assume that g is monic and $\gcd(f,\, g) = 1$. Let $g = (x - c_1) \cdots (x - c_m)$ be the factorisation of g over its minimal splitting field K. Observe that, because g is square-free, each of these (irreducible) factors is distinct. By *Theorem 2.1*, there exists a $b \in \mathbb{Q}[x]$ and $a_1, \ldots, a_m \in \mathbb{Q}$ such that

$$\frac{f}{g} = b + \sum_{i=1}^{m} \frac{a_i}{x - c_i}. \tag{9}$$

Therefore

$$\int \frac{f}{g} = \int b + \sum_{i=1}^{m} a_i \log(x - c_i), \tag{10}$$

which lies in $K(x, \log(x - c_1), \ldots, \log(x - c_m))$.

2.2 Hermite's Method

Having found a complete procedure for rational integration when the denominator is square-free, it remains for us to consider the integrals of those rational functions

$$\frac{f}{g}, \qquad \gcd(f,\, g) = 1,\ g \text{ monic and } \textit{not} \text{ square-free.} \tag{11}$$

Theorem 2.3 (Square-free Decomposition)**.** *Let F be a field and $p \in F[x] - \{0\}$. There exist unique monic polynomials $\{q_i\}_{i=1}^m$ in $F[x] - \{0\}$ and $r \in F$ such that each q_i is square-free,* $\gcd(q_i, q_j) = 1$ *whenever $i \neq j$, and*

$$p = r \prod_{i=1}^{m} q_i^i. \tag{12}$$

We call this the square-free decomposition *of p.*

Example 2.8. We use Magma to find the square-free decomposition of $2x^7 - 8x^6 + 14x^5 - 40x^4 + 82x^3 - 76x^2 + 120x - 144$ over $\mathbb{Q}[x]$.

```
1 > P<x> := PolynomialRing(RationalField());
2 > SquarefreeFactorisation(2*x^7 - 8*x^6 + 14*x^5 - 40*x^4 + 82*x
      ^3 - 76*x^2 + 120*x - 144);
3 [
4     <x^2 + x + 3, 2>,
5     <x - 2, 3>
6 ]
```

On Line 1 we declare `P` to be $\mathbb{Q}[x]$, using angled brackets to bind `x` as the symbol of the polynomial ring. Magma returns the square-free decomposition on Line 3 - Line 6, represented as a sequence of (*square-free factor*, *multiplicity*) pairs. Here, Magma uses angled brackets to represent n-tuples. We interpret this output as $(x^2 + x + 3)^2(x - 2)^3$. ■

In a manner analogous to that of *Theorem 2.1*, we can find a partial fraction decomposition of the $\frac{f}{g}$ in Equation (11) using the *square-free decomposition* of g instead of the factorisation into irreducibles.

Theorem 2.4 (Square-free Partial Fraction Decomposition)**.** *Suppose F is a field and $f, g \in F[x] - \{0\}$ with $\gcd(f, g) = 1$. Without loss of generality, assume that g is monic such that $g = \prod_{i=1}^{m} q_i^i$ is the square-free decomposition of g. Then there exist polynomials b and r_{ij}, $1 \leq j \leq i$, $1 \leq i \leq m$ such that*

$$\frac{f}{g} = b + \sum_{i=1}^{m} \sum_{j=1}^{i} \frac{r_{ij}}{q_i^j} \tag{13}$$

and $\deg r_i < \deg q_i$ for all $i = 1, \ldots, m$.

Example 2.9. Let $f = 4x^7 - 16x^6 + 28x^5 - 351x^3 + 588x^2 - 738$ and $g = 2x^7 - 8x^6 + 14x^5 - 40x^4 + 82x^3 - 76x^2 + 120x - 144$. We continue the Magma session from *Example 2.8* to find the *square-free partial fraction decomposition* of $\frac{f}{g}$ over $\mathbb{Q}(x)$.

```
7  > f := 4*x^7 - 16*x^6 + 28*x^5 - 351*x^3 + 588*x^2 - 738;
8  > g := 2*x^7 - 8*x^6 + 14*x^5 - 40*x^4 + 82*x^3 - 76*x^2 + 120*x
       - 144;
9  > GCD(f, g);
10 1
11 > sfpf := SquarefreePartialFractionDecomposition(f/g);
12 > sfpf;
13 [
14     <1, 1, 2>,
15     <x^2 + x + 3, 2, 45*x + 45/2>,
16     <x - 2, 3, -5>
17 ]
```

We initialise the variables `f` and `g` on Line 7 and Line 8 respectively. On Line 9, we check that $\gcd(f, g) = 1$ as required. The output `sfpf` of the square-free partial fraction decomposition from Line 13 to Line 17 is a sequence of (*denominator*, *multiplicity*, *numerator*) triples. We interpret this output as

$$2 + \frac{45x + \frac{45}{2}}{(x^2 + x + 3)^2} + \frac{-5}{(x - 2)^3}. \tag{14}$$

We can check that the square-free partial fraction decomposition is correct by taking the sum $\sum a/(b^c)$ over the $(a,\, b,\, c)$ triples in `sfpf`.

```
> f/g eq &+[ tm[3]/(tm[1]^tm[2]) : tm in sfpf];
true
```

The `&+` operator sums the sequence it takes as an argument. We use square brackets to index into a triple (i.e. `<a, b, c>[1]` is `a` and `<a, b, c>[3]` is `c`). ■

By *Theorem 2.4*, each term in the *square-free partial fraction decomposition* of the $\frac{f}{g}$ in Equation (11) is of the form

$$\frac{r}{q^j}, \qquad \deg r < \deg q,\ q \text{ square-free.} \tag{15}$$

We derive a recursive algorithm called *Hermite Reduction*, attributed to the 19th-century French mathematician *Charles Hermite*. Each step of this algorithm either

(i) reduces the problem $\int \frac{r}{q^j}$ to $\int \frac{r}{q^{j-1}}$; or

(ii) solves the integral outright.

The base case for the algorithm is $j = 1$, in which case the denominator is square-free, so we may factorise the denominator over its minimal splitting field and apply integration by partial fraction decomposition.

Proposition 2.2. *Let F be a field. A polynomial $p \in F[x]$ is square-free if and only if $\gcd(p,\, \frac{d}{dx}p) = 1$.*

Proof. Let $p = \prod_{i=1}^{k} q_i^{n_i}$ be the factorisation of f into irreducibles. Then

$$\frac{d}{dx}p = \sum_{i=1}^{k} n_i q_i' q_i^{n_i - 1} \prod_{j \neq i} q_j^{n_j}. \tag{16}$$

Because each q_i is irreducible and $\deg q_i' < \deg q_i$, we must have $\gcd(q_i,\, q_i') = 1$. Moreover, any two distinct irreducible factors have greatest common divisor 1, so some irreducible factor q_i of p divides p' if and only if $n_i > 1$. The result follows immediately. □

Theorem 2.5 (Bèzout's Lemma)**.** *Let F be a field and $p, q, g \in F[x]$. There exist polynomials $a, b \in F[x]$ such that $ap + bq = g$ if and only if $\gcd(p, q) \mid g$. Moreover, if any a, b exist, there are particular solutions with $\deg a < \deg q$ and $\deg b < \deg p$.*

Let us assume that $j > 1$ in $\int \frac{r}{q^j}$ from Equation (15). Since q is square-free, we may appeal to *Proposition 2.2* to find that $\gcd(q, q') = 1$. By *Theorem 2.5*, there exist $s, t \in \mathbb{Q}[x]$ such that $sq + tq' = r$. Then

$$\int \frac{r}{q^j} = \int \frac{s}{q^{j-1}} + \int \frac{tq'}{q^j}. \tag{17}$$

It remains for us to find a way to reduce the power of q in the denominator of the final term of Equation (17).

Proposition 2.3 (Integration by Parts)**.** *Let $u, v \in \mathbb{Q}[x]$ and let (G, D) be as in* Definition 2.1. *Then $\int u'v \in G$ if and only if $uv - \int uv' \in G$. Moreover, whenever this happens, we have $\int u'v = uv - \int uv'$.*

Proof. The result follows immediately from the observation that $u'v = (uv)' - uv'$. □

Taking $u = t$ and $v = -\dfrac{1}{(j-1)q^{j-1}}$ and applying *Proposition 2.3* in Equation (17), we have

$$\begin{aligned}\int \frac{r}{q^j} &= \int \frac{s}{q^{j-1}} + \frac{-t/(j-1)}{q^{j-1}} + \int \frac{t'/(j-1)}{q^{j-1}} \\ &= \frac{-t/(j-1)}{q^{j-1}} + \int \frac{s + t'/(j-1)}{q^{j-1}}. \end{aligned} \tag{18}$$

This completes our derivation of *Hermite Reduction*.

Theorem 2.6 (Existence of Rational Integrals)**.** *Let $\frac{p}{q} \in \mathbb{Q}(x)$ with $\gcd(p, q) = 1$ and q monic. There exists a differential extension (G, D) as in* Definition 2.1 *and a $g \in G$ such that $g' = \frac{p}{q}$.*

Proof sketch. If $q = 1$ then apply *Example 2.3*. Otherwise, let $q = \prod_{i=1}^{m} s_i^i$ be the square-free decomposition of q. Proceed by induction on m, the highest multiplicity of a square-free factor in the denominator. The base case is when $m = 1$ and we may apply the method of partial fractions. The inductive step follows using *Hermite Reduction*. □

Example 2.10. We compute

$$\int \frac{f}{g} = \int \frac{4x^7 - 16x^6 + 28x^5 - 351x^3 + 588x^2 - 738}{2x^7 - 8x^6 + 14x^5 - 40x^4 + 82x^3 - 76x^2 + 120x - 144}. \tag{19}$$

In *Example 2.9* we find the square-free partial fraction decomposition

$$\frac{f}{g} = 2 + \frac{45x + \frac{45}{2}}{(x^2 + x + 3)^2} + \frac{-5}{(x-2)^3}. \tag{20}$$

We use Magma to compute the numerators of Equation (18) for $\displaystyle\int \frac{-5}{(x-2)^3}$.

```
20 > r := -5;
21 > q := x - 2;
22 > j := 3;
23 > _, s, t := XGCD(q, Derivative(q));
24 > s *:= r; t *:= r;
25 > -t/(j - 1), s + Derivative(t)/(j - 1);
26 5/2
27 0
```

On Line 23 we use the *extended GCD algorithm* to compute s and t such that $sq+tq' = \gcd(q,\, q')$. We know that this GCD is 1, so we ignore the first output of `XGCD` using `_` (instead of binding to a variable). As s, t is a solution to $sq + tq' = 1$, we multiply both by r on Line 24 so that `r` and `s` are solutions to $sq + tq' = r$. We interpret the output on Line 26 and Line 27 as

$$\int \frac{-5}{(x-2)^3} = \frac{\frac{5}{2}}{(x-2)^2} + \int 0. \tag{21}$$

We now compute the numerators of Equation (18) for $\displaystyle\int \frac{45x + \frac{45}{2}}{(x^2+x+3)^2}$.

```
28 > r := 45*x + 45/2;
29 > q := x^2 + x + 3;
30 > j := 2;
31 > _, s, t := XGCD(q, Derivative(q));
32 > s *:= r; t *:= r;
33 > -t/(j - 1), s + Derivative(t)/(j - 1);
34 90/11*x^2 + 90/11*x + 45/22
35 0
```

We interpret the output on Line 34 and Line 35 as

$$\int \frac{45x + \frac{45}{2}}{(x^2+x+3)^2} = \frac{\frac{90}{11}x^2 + \frac{90}{11}x + \frac{45}{22}}{x^2+x+3} + \int 0 \tag{22}$$

Therefore

$$\int \frac{4x^7 - 16x^6 + 28x^5 - 351x^3 + 588x^2 - 738}{2x^7 - 8x^6 + 14x^5 - 40x^4 + 82x^3 - 76x^2 + 120x - 144} = 2x + \frac{\frac{5}{2}}{(x-2)^2} + \frac{\frac{90}{11}x^2 + \frac{90}{11}x + \frac{45}{22}}{x^2+x+3}. \tag{23}$$

∎

2.3 The Rothstein-Trager Theorem

The splitting field we compute for integration by partial fractions is computationally expensive (see Herstein [6]) and sometimes unnecessary. Moreover, we may need to work in an algebraic extension of $\mathbb{Q}$ after we have factorised the denominator into linear factors. There is an improvement given by *Rothstein* and *Trager* which expresses the integral in an extension field G as in *Definition 2.1* which has both the *smallest constant field* and the *fewest logarithms.*

Definition 2.2 (Resultant)**.** Let F be a field and $a, b \in F[x]$ with $\deg a = m$ and $\deg b = n$. Let $\mathcal{P}_k$ be the F-vector space $\{p \in F[x] \mid p = 0 \text{ or } \deg p < k\}$. The *resultant* $\operatorname{res}(a, b)$ of a and b is the determinant of the linear map $\varphi : \mathcal{P}_n \times \mathcal{P}_m \to \mathcal{P}_{m+n}$ given by $\varphi : (p, q) \mapsto ap + bq$.

If the coefficients of the polynomials in *Definition 2.2* come from a system which is not a field, but does satisfy all the field axioms except (F6) (they are from a *commutative ring with unit*), the definition still holds because we can still talk about linear maps and determinants. The resultant is fundamental to the differences between the roots of two polynomials. One can show that

$$\operatorname{res}(a, b) = \operatorname{lc}(a) \operatorname{lc}(b) \prod_{\substack{1 \le i \le m \\ 1 \le j \le n}} (\lambda_i - \mu_j), \tag{24}$$

where $\operatorname{lc}(p)$ is the *leading coefficient* of p (that is, the coefficient corresponding to the highest power of x in p) and λ_i, μ_j are the respective roots of a and b over their splitting fields (counted with their multiplicities).

Theorem 2.7 (Rothstein-Trager)**.** *Let K be a finitely generated algebraic extension of $\mathbb{Q}$ and $\frac{a}{b} \in K(x)$ with* $\gcd(a, b) = 1$ *and b monic and square-free. Let* $R(z) = \operatorname{res}_x(a - zb', b) \in K[z]$ *and* $\{c_i\}_{i=1}^m$ *be the* distinct *roots of R over its minimal splitting field K^*. Then*

$$\int \frac{a}{b} = \sum_{i=1}^{m} c_i \log(\gcd(a - c_i b', b)), \tag{25}$$

which lies in $K^*(x, \log(\gcd(a - c_1 b', b)), \ldots, \log(\gcd(a - c_m b', b)))$.

We note that the algebraic extension field K^* in *Theorem 2.7* is always the *smallest possible* needed to express the integral.

Example 2.11. We use *Theorem 2.7* to find $\displaystyle\int \frac{4x^2 + 2x - 4}{x^3 + x^2 - 2x - 2}$ in Magma.

```
36 > a := 4*x^2 + 2*x - 4;
37 > b := x^3 + x^2 - 2*x - 2;
38 > PP<y, z> := PolynomialRing(RationalField(), 2);
39 > h := hom< P -> PP | y >;
40 > r := Resultant(h(a) - z*h(Derivative(b)), h(b), 1);
41 > r;
42 8*z^3 - 32*z^2 + 40*z - 16
43 > rts:= Roots(UnivariatePolynomial(r));
44 > rts
45 [ <1, 2>, <2, 1> ]
```

On Line 38, we declare a new multivariate polynomial ring `PP` with symbols `y` and `z`. We can lift univariate polynomials from `P` into `PP` by changing the symbol `x` to `y`. This is exactly what the homomorphism `h` we declare on Line 39 does. On Line 40 we compute a resultant inside `PP` with respect to the first symbol `y`. Magma gives the roots of `r` on Line 45 as a sequence of (*root*, *multiplicity*) pairs.

We can represent the sum of constants multiplied by logarithms we get from *Theorem 2.7* as a sequence of (*coefficient*, *logarithm argument*) pairs.

```
46 > [<rt[1], GCD(a - rt[1]*Derivative(b), b)> : rt in rts];
47 [
48     <1, x^2 - 2>,
49     <2, x + 1>
50 ]
```

We interpret the output from Line 47 to Line 50 as

$$\int \frac{4x^2 + 2x - 4}{x^3 + x^2 - 2x - 2} = \log(x^2 - 2) + 2\log(x + 2). \tag{26}$$

Observe that if we had applied the method of partial fractions then we would have needed to work in the constant field $\mathbb{Q}(\sqrt{2})$, only to simplify the integral back to the one in Equation (26) and remove the need for any irrational algebraic numbers. ∎

3 Transcendental Integration

We briefly discuss the broader field of *transcendental integration*, stating some of the key definitions and theorems and exploring some interesting examples. Geddes [5] and Bronstein [3] are both excellent references for further reading.

Definition 3.1 (Elementary Function Field)**.** Let $K = \mathbb{Q}(\alpha_1, \ldots, \alpha_m)$ be a constant differential field (with differential operator $'$) where each α_i is algebraic over $\mathbb{Q}$. Let x be a transcendental symbol over K satisfying $x' = 1$ (in the differential field $(K(x), ')$). In the differential extension field $K(x, \theta_1, \ldots, \theta_n)$, we say each θ_j $(j = 1, \ldots, n)$ is *elementary* over $K(x, \theta_1, \ldots, \theta_{j-1})$ if θ_j is *transcendental* over $K(x, \theta_1, \ldots, \theta_{j-1})$ and either:

(i) $\theta_j' = u'/u$ for some $u \in K(x, \theta_1, \ldots, \theta_{j-1})$ (in which case we say θ_j is *logarithmic*), or

(ii) $\theta_j'/\theta_j = u'$ for some $u \in K(x, \theta_1, \ldots, \theta_{j-1})$ (in which case we say θ_j is *exponential*).

If each θ_j is elementary, then we say that $K(x, \theta_1, \ldots, \theta_{j-1})$ is an *elementary function field.*

Definition 3.2 (Elementary Extension)**.** Let F be an elementary function field. We say G is an *elementary extension* of F if G is an elementary function field, G is a differential extension of F, and G has the same constant field as F.

Definition 3.3 (Integration)**.** Let F be an elementary function field and $f \in F$. If there exists a finitely and explicitly generated elementary extension G of F and a $g \in G$ with $g' = f$, then we say g is the *elementary antiderivative* or *elementary integral* of f and write $\int f = g$. If no such g or G exist, we say that f has no elementary integral.

Although we require that an elementary extension has the same constant field in *Definition 3.2*, in practice we can *extend* the constant field of an elementary function field with additional algebraic numbers as needed. This allows us to work in a constant field which is as small as possible. Moreover, it is often not obvious which algebraic numbers will be needed to express an integral when deciding the elementary function field in which to express the integrand.

Notice that *Definition 2.1* is a special case of *Definition 3.3.* Risch [7] was the first to give a complete procedure for integration in elementary function fields. We briefly discuss this procedure, called the *Risch Procedure* or *Risch Algorithm*, in this section. If F is an elementary function field and θ is elementary over F, the algorithm finds an elementary integral of an element of $F(\theta)$ or proves that there is none by recursively performing integration in F.

3.1 Liouville's Principle

Liouville's Principle is a theorem which is absolutely fundamental for integrating the elements of elementary function fields. We prove two theorems which are useful in the integration of polynomials in $F[\theta]$ where F is an elementary function field and θ is transcendental and logarithmic or exponential over F.

Theorem 3.1. *Let F be an elementary function field and θ be transcendental and logarithmic over F, with $\theta' = \frac{u'}{u}$ for some $u \in F$. If $f \in F[\theta]$ with $\deg f > 0$, then:*

(i) $f' \in F[\theta]$.

(ii) *If the leading coefficient of f is constant, then* $\deg f' = \deg f - 1$.

(iii) *If the leading coefficient of f is* not *constant, then* $\deg f' = \deg f$.

Proof. Let $f = \sum_{i=0}^{m} f_i\theta^i$ for some $m \in \mathbb{N}$ and $f_0, \ldots, f_m \in F$. Then

$$\begin{aligned} f' &= \sum_{i=0}^{m} (f_i\theta^i)' \\ &= \sum_{i=0}^{m} \left(f_i'\theta^i + i f_i \frac{u'}{u}\theta^{i-1} \right) \\ &= f_m'\theta^m + \sum_{i=0}^{m-1} \left(f_i' + (i+1) f_{i+1}\frac{u'}{u} \right)\theta^i \end{aligned} \tag{27}$$

As $f_m' \in F$ and $f_i' + (i+1)f_{i+1}\frac{u'}{u} \in F$ for $i = 0, \ldots, m-1$, we have (i). Clearly we have (ii) also. It remains to show that if f_m is constant, then the coefficient of θ^{m-1} in f' is non-zero. Suppose not. Then $f_{m-1}' + m f_m \frac{u'}{u} = 0$, so $\frac{u'}{u} = \frac{-1}{m f_m} \cdot f_{m-1}'$. Integrating both sides, we have $\theta = \frac{-1}{m f_m} f_{m-1} \in F$, a contradiction (since θ is transcendental). Therefore (iii). □

Theorem 3.2. *Let F be an elementary function field and θ be transcendental and exponential over F, with $\theta'/\theta = u'$ for some $u \in F$.*

(i) *For all $g \in F - \{0\}$ and $n \in \mathbb{Z} - \{0\}$, there exists a $h \in F - \{0\}$ such that $(g\theta^n)' = h\theta^n$.*

(ii) *If* $f \in F[\theta]$ *with* $\deg f > 0$*, then* $f' \in F[\theta]$ *and* $\deg f' = \deg f$.

Proof. Observe that $(g\theta^n)' = g'\theta^n + ngu'\theta \cdot \theta^n = (g' + ngu')\theta^n$. Suppose for the sake of contradiction that $h = g' + ngu' = 0$. Then $(g\theta^n)' = 0$, so $g\theta^n \in F$. This contradicts our assumption that θ is transcendental, so (i). (ii) follows immediately. □

Theorem 3.3 (Liouville's Principle)**.** *Let* (F, D) *be a differential field and* $f \in F$*. If there exists an elementary extension* (G, E) *of* F *and a* $g \in G$ *such that* $Eg = f$*, then there exist* $v_0, \ldots, v_m \in F$ *and* $c_1, \ldots, c_m$ *in the constant field of* G *such that*

$$f = Dv_0 + \sum_{i=1}^{m} c_i \frac{Dv_i}{v_i}. \tag{28}$$

Proof sketch. Let $G = F(\theta_1, \ldots, \theta_m)$ and proceed by induction on m. The base case is $G = F$, in which case the integral lies in F, so the theorem clearly holds. For the inductive step, let $G = F(\theta_1, \ldots, \theta_k, \theta_{k+1})$ and suppose the theorem holds in $F(\theta_1, \ldots, \theta_k)$. Write $g = \frac{a}{b}$ for some non-zero $a, b \in F(\theta_1, \ldots, \theta_k)[\theta_{k+1}]$ with $\gcd(a, b) = 1$. If θ_{k+1} is logarithmic with $E\theta = \frac{Eu}{u}$ for some $u \in F$, then there is nothing to do. If $u \notin F$, we use *Theorem 3.1* to show that, if g depends on θ_{k+1}, this dependence would fail to disappear when taking the derivative $Eg = f$. If θ_{k+1} is exponential, we similarly use *Theorem 3.2* to show that any dependence of θ_{k+1} in g fails to disappear under the derivative. □

In the field of symbolic integration (or *integration in finite terms*), *Liouville's Principle* is often stated in terms of new symbols which are *algebraic* (for example $\sqrt{x}$ over $\mathbb{Q}(x)$). Other research, such as that of Singer et al. [10], gives generalisations to other transcendentals, such as non-elementary exponential integrals $\int u' \exp(-u^2)$ and the logarithmic integral $\int \frac{1}{\log x}$.

3.2 Logarithmic Integration

Let F be an elementary function field and θ elementary and logarithmic over F. We consider the integration of functions $\frac{f}{g} \in F(\theta)$, where $f, g \in F[\theta]$. Similarly to the rational case, Geddes [5] shows that there is a variation of *Hermite's Method* that we can apply to reduce the problem of $\int \frac{f}{g}$ to $\int p + \int \frac{a}{b}$, where $p, a, b \in F[\theta]$ and b is monic and square-free.

Theorem 3.4 (Rothstein-Trager — Logarithmic Case)**.** *Let* F *be an elementary function field and with constant field* K*. Let* θ *be elementary and logarithmic over* F *and suppose that* $F(\theta)$ *has* *the same constant field* K. *Let* $\frac{a}{b} \in F(\theta)$ *where* $a, b \in F[\theta]$ *with* $\gcd(a,\, b) = 1$ *and* b *monic and square-free. Let* $R = \operatorname{res}_\theta(a - zb',\, b) \in F[z]$.

(i) $\int \frac{a}{b}$ *is elementary if and only if all of the roots of* R *are constants (equivalently,* $R/\operatorname{lc}(R) \in K[z]$*).*

(ii) *If* $\int \frac{a}{b}$ *is elementary then*

$$\frac{a}{b} = \sum_{i=1}^{m} c_i \frac{v_i'}{v_i}, \tag{29}$$

where $\{c_i\}_{i=1}^m$ *are the distinct roots of* R *over its splitting field and* $v_i = \gcd(a - c_i b',\, b)$ *for each* $i = 1, \ldots, m$.

Example 3.1. Let $F = \mathbb{Q}(x)$, $\theta = \log x$ and consider

$$\int \frac{1}{x \log x}. \tag{30}$$

The integrand here satisfies the assumptions of *Theorem 3.4*, wherein we calculate the resultant $\operatorname{res}_\theta(1 - z(\theta + 1),\, x\theta) = xz - x$. This polynomial has a single root at $z = 1$, which is constant. Therefore

$$\int \frac{1}{x\theta} = \log(\gcd(1 - (\theta + 1),\, x\theta)) = \log\theta = \log(\log x) \quad \in F(\theta,\, \log\theta). \tag{31}$$

■

Example 3.2. Let $F = \mathbb{Q}(x)$, $\theta = \log x$ and consider

$$\int \frac{1}{\theta}. \tag{32}$$

We may apply *Theorem 3.4* and calculate the resultant $R = \operatorname{res}_\theta(1 - z \cdot \frac{1}{x},\, \theta) = 1 - z\frac{1}{x}$. This polynomial has a single root at $z = x$, which is not constant. Therefore $\frac{1}{\theta}$ has no elementary integral. ■

Unlike in rational integration, integrating polynomials in $F[\theta]$ is not trivial. Suppose $p = \sum_{i=0}^m p_i\theta^i$ is such a polynomial, with $m \in \mathbb{N}$ and $p_0, \ldots, p_m \in F$, $p_m \neq 0$. By *Theorem 3.3*, if p has an elementary integral, it is of the form

$$\int \left(\sum_{i=0}^{m} p_i \theta^i \right) = v_0 + \sum_{j=1}^{n} c_j \log v_j \tag{33}$$

for some constants $c_1, \ldots, c_n$ and $v_0, \ldots, v_n \in F(\theta)$. Although we will not go through the details, one can use *Theorem 3.1* to show that $v_1, \ldots, v_n \in F$ and $v_0 = \sum_{i=0}^{m+1} q_i\theta^i$ with $q_0, \ldots, q_{m+1} \in F$ such that q_{m+1} is constant (and possibly 0) and, if q_{m+1} is 0, then q_m is non-zero. Therefore

$$\sum_{i=0}^{m} p_i\theta^i = q_{m+1}\theta^{m+1} + \sum_{i=1}^{m}(q_i' + (i+1)q_{i+1}'\theta')\theta^i + q_1\theta' + q_0 + \sum_{j=1}^{n} c_j\frac{v_j'}{v_j}. \tag{34}$$

This gives us all the tools we need to solve for each coefficient of θ in the right-hand-side of Equation (34). Moreover, one can show that $\int p$ is elementary if and only if we can find solutions $q_i = a_i + b_i\theta$ for $a_i, b_i \in F$ with b_i constant (and possibly 0) for each $i = 1, ..., m$.

Example 3.3. Let $F = \mathbb{Q}(x)$ and $\theta = \log x$. Consider $\int(2\theta + 2)$. From Equation (34) we have

$$\int(2\theta + 2) = q_2\theta^2 + q_1\theta + \bar{q}_0 \tag{35}$$

for some $q_2, q_1, \bar{q}_0 \in \mathbb{Q}(x)$ with q_2 constant. Taking the derivative of Equation (35), we find

$$2\theta + 2 = (q_1' + 2q_2\theta')\theta + q_1' + \bar{q}_0. \tag{36}$$

Equating the coefficients of θ in (36), we have $2 = q_1' + 2q_2\theta'$. It follows that

$$q_1 = \int 2 - 2q_2\int\theta' = 2x - 2q_2\theta, \tag{37}$$

so $q_2 = 0$ and $q_1 = 2x$. Finally, equating the coefficients of θ^0 in (36) gives us $2 = q_1' + \bar{q}_0 = 2 + \bar{q}_0$, so $\bar{q}_0 = 0$. Therefore $\int(2\theta + 2) = 2x\theta \in F(\theta)$. ■

3.3 Exponential Integration

Let F be an elementary function field and θ elementary and exponential over F. Unlike in the logarithmic and rational cases, a polynomial $p \in F[\theta]$ has $\gcd(p,\, p') = 1$ if and only if p is square-free *and* $\theta \nmid p$. As a result of this, the variation of *Hermite's Method* for exponentials reduces the problem of integrating $\frac{f}{g} \in F(\theta)$ to $\int p + \int \frac{a}{b}$ where $p \in F[\theta,\, \theta^{-1}]$ is an *extended polynomial* and $a, b \in F[\theta]$ with $\gcd(a,\, b) = 1$, b monic and square-free and $\theta \nmid b$.

Theorem 3.5 (Rothstein-Trager — Exponential Case)**.** *Let F be an elementary function field with constant field K. Let θ be elementary and exponential over F with $\theta'/\theta = u' \in F$ and suppose that $F(\theta)$ has the same constant field K. Let $\frac{a}{b} \in F(\theta)$ where $a, b \in F[\theta]$ with $\gcd(a,\, b) = 1$, $\deg a < \deg b$, $\theta \nmid b$ and b monic and square-free. Let $R = \mathrm{res}_\theta(a - zb',\, b) \in F[z]$.*

(i) $\int \frac{a}{b}$ *is elementary if and only if all the roots of* R *are constants (equivalently,* $R/\operatorname{lc}(R) \in K[z]$*).*

(ii) *If* $\int \frac{a}{b}$ *is elementary then*

$$\frac{a}{b} = g' + \sum_{i=1}^{m} c_i \frac{v_i'}{v_i}, \tag{38}$$

where $\{c_i\}_{i=1}^m$ *are the distinct roots of* R *over its splitting field,* $v_i = \gcd(a - c_i b', b)$ *for each* $i = 1, \ldots, m$*, and* $g \in F(c_1, \ldots, c_m)$ *is given by*

$$g' = -u' \sum_{i=1}^{m} c_i \deg v_i. \tag{39}$$

Example 3.4. Let $F = \mathbb{Q}(x)$ and consider

$$\int \frac{\exp 2x}{\exp 2x + \exp x + 1}. \tag{40}$$

First note that $\exp 2x$ is *not* transcendental over $\mathbb{Q}(x, \exp x)$ as $\exp 2x = (\exp x)^2$. Therefore we may write the integral as

$$\int \frac{\theta^2}{\theta^2 + \theta + 1}, \tag{41}$$

where $\theta = \exp x$. Applying Euclidean division, we find

$$\frac{\theta^2}{\theta^2 + \theta + 1} = 1 + \frac{-\theta - 1}{\theta^2 + \theta + 1}. \tag{42}$$

It is trivial for us to find $\int 1 = x$, so it remains for us to compute $\int \frac{-\theta-1}{\theta^2+\theta+1}$. This integrand satisfies the conditions for *Theorem 3.5*, wherein we calculate the resultant $R = 3z^2 - 3z + 1$. We compute

$$\int \frac{-\theta - 1}{\theta^2 + \theta + 1} = \alpha \log(\theta - 3\alpha + 2) + (1 - \alpha) \log(\theta + 3\alpha - 1) - x, \tag{43}$$

where α and $1 - \alpha$ are the roots of the R over its splitting field. Therefore

$$\int \frac{\theta^2}{\theta^2 + \theta + 1} = \alpha \log(\theta - 3\alpha + 2) + (1 - \alpha) \log(\theta + 3\alpha - 1). \tag{44}$$

■

We now consider the integration of extended polynomials $p = \sum_{i=-k}^{m} p_i \theta^i \in F[\theta, -\theta]$ with $p_m, p_{-k} \neq 0$. By *Theorem 3.3*, if p has an elementary integral, it is of the form

$$\int \left(\sum_{i=-k}^{m} p_i \theta^i \right) = v_0 + \sum_{i=1}^{n} c_j \log v_j \tag{45}$$

for some constants $c_1, \ldots, c_n$ and $v_0, \ldots, v_n \in F$. One can use *Theorem 3.2* and the fact that $\gcd(f, f') = 1$ for some $f \in F[\theta]$ only when f is square-free and $\theta \nmid f$ to show that we must have $v_1, \ldots, v_n \in F$ and $v_0 = \sum_{i=-k}^{m} q_i \theta^i$ for some $q_{-k}, \ldots, q_m \in F$ with $q_{-k}, q_m \neq 0$. Therefore

$$p = \sum_{i=-k}^{m} p_i \theta^i = \sum_{i=-k}^{m} (q_i' + iu' q_i)\theta^i + \sum_{j=1}^{n} c_j \frac{v_j'}{v_j} \tag{46}$$

where $u' = \theta'/\theta \in F$.

By equating powers of θ in Equation (46), we find that solving $\int p$ is equivalent to solving a system of equations. For the term corresponding to θ^0, we see that if $\int p$ is elementary then $q_0 + \sum_{j=1}^{n} c_j \log v_j = \int p_0$. If $\int p_0$ is not elementary, we may conclude that $\int p$ is not elementary. For all other powers of θ, we need to solve the ordinary differential equation $p_i = q_i' + iu' q_i$ for $q_i \in F$.

It seems that we have turned the problem of integration into the much harder problem of solving an ordinary differential equation (ODE). However, because the ODEs are of the particular form $y' + fy = g$ and we know the specific elementary field that solutions must come from, this problem is solvable and is named the *Risch differential equation*. Risch [7] was the first to give a (incomplete) procedure to solve this differential equation in his first paper on integration in finite terms. Rothstein [9] and Davenport [4] built on his work to develop complete algorithms, but Bronstein [2] notices an error in their work, which he addresses in his algorithm. Moreover, his algorithm explicitly gives the denominator of a solution y, if any exists. Bronstein's paper [2] is an excellent reference for further reading.

Example 3.5. Let $F = \mathbb{Q}(x)$ and $\theta = \exp x$. Consider

$$\int \left(\theta + x\theta + \frac{2}{x}\theta^2 - \frac{1}{x^2}\theta^2 \right). \tag{47}$$

From Equation (46), we have

$$\int \left(\theta + x\theta + \frac{2}{x}\theta^2 - \frac{1}{x^2}\theta^2 \right) = q_1\theta + q_2\theta^2 \tag{48}$$

(if any elementary integral exists). Differentiating both sides, we find

$$(1 + x)\theta + \left(\frac{2}{x} - \frac{1}{x^2} \right) \theta^2 = (q_1 + q_1')\theta + (2q_2 + q_2')\theta^2. \tag{49}$$

Therefore we seek $q_1, q_2 \in \mathbb{Q}(x)$ such that $1 + x = q_1 + q_1'$ and $\frac{2}{x} - \frac{1}{x^2} = 2q_2 + q_2'$. By inspection we have that $q_1 = x$. For q_2, notice that $\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$, so $q_2 = \frac{1}{x}$ is a solution. Therefore

$$\int \left(\theta + x\theta + \frac{2}{x}\theta^2 - \frac{1}{x^2}\theta^2\right) = x\theta + \frac{1}{x}\theta^2. \tag{50}$$

■

Example 3.6. Let $F = \mathbb{Q}(x)$ and $\theta = \exp(x^2)$. Then $\int \theta$ is elementary if and only if there is a solution $q \in F$ to the *Risch differential equation* $1 = q' + 2xq$. Suppose such a solution exists. Clearly the solution must be non-zero, so we can write $q = \frac{a}{b}$ for $a, b \in \mathbb{Q}[x] - \{0\}$ with $\gcd(a, b) = 1$. Then

$$1 = (ab^{-1})' + 2xab^{-1} = a'b^{-1} + ab'b^{-2} + 2xab^{-1}, \tag{51}$$

so

$$b^2 = a'b - ab' + 2xab = b(2xa - a') - ab'. \tag{52}$$

Clearly $b \mid b^2$ and $b \mid b(2x - a')$, so $b \mid ab'$ also. In particular, we cannot have $b' = 0$, so $\deg b > \deg b' \geq 0$. As $\gcd(a, b) = 1$, we must have $b \mid b'$. But then $\deg b \leq \deg b'$, a contradiction. Therefore there is no solution $q \in F$ to $1 = q' + 2xq$, so $\int \theta$ cannot be elementary. ■

3.4 The Risch Structure Theorem

Recall that the Risch procedure for integration assumes that each generator θ_i of an elementary function field $K(x, \theta_1, \ldots, \theta_n)$ is *transcendental* over $K(x, \theta_1, \ldots, \theta_{i-1})$. We see in *Example 3.6* that this assumption does not always hold; indeed, it is often not trivial to decide if a new generator is transcendental. Risch notes this in his first paper [7] on integration, where it is left as an open problem. Risch himself solved this with a theorem in his later paper [8], which has since become known as the *Risch Structure Theorem.*

Theorem 3.6 (Risch Structure Theorem)**.** *Let $F = K(x, \theta_1, \ldots, \theta_n)$ be an elementary function field.*

(i) $\log f$ *with $f \in F - K$ is transcendental over F if and only if there is no product combination $f^k \prod u_j^{k_j} \in K$, where $k, k_j \in \mathbb{Z}$, $k \neq 0$, and $\log u_j$ are the logarithms appearing amongst $\theta_1, \ldots, \theta_n$.*

(ii) $\exp g$ *with* $a \in F - K$ *is transcendental over* F *if and only if there is no linear combination* $g + \sum c_i w_i \in K$*, where* $c_i \in \mathbb{Q}$ *and* $\exp w_i$ *are the exponentials appearing amongst* $\theta_1, \cdots, \theta_m$.

Computationally, it is very efficient to check that the assumptions of *Theorem 3.6* have been satisfied. In the exponential case, notice that $g + \sum c_i w_i \in K$ if and only if there exist $c_i \in K$ with $g' + \sum c_i w_i' = 0$, so we need only solve a linear system. For the logarithmic case, let $h = f^k \prod u_j^{k_j}$. For all $k, k_j \in \mathbb{Z}$ with $k \neq 0$ and $c \in K$, we have $c = h \iff 0 = h' \iff 0 = \frac{h'}{h}$. Observe that

$$\frac{h'}{h} = k\frac{f'}{f} + \sum k_j \frac{u_j'}{u_j}, \tag{53}$$

so we see that we need only solve a linear system to check that a new logarithm is transcendental.

4 Conclusion

This paper presented the theory of rational integration in a way that is approachable to readers who are not necessarily familiar with any abstract algebra. We expressed the existing methods for rational integration in a way that required very little of the language of abstract algebra. We went on to discuss the method of *Rothstein-Trager*, which reduces the amount of work to generate splitting fields to the minimum amount needed, thus removing a not-insignificant barrier to implementing a fast and effective algorithm for rational integration.

Moreover, throughout the discussion of rational integration, we used the Magma computer algebra system to compute examples in a way that could be extended to a full implementation of rational integration.

Finally, we briefly discussed the broader field of transcendental integration, most notably discussing a proof of *Liouville's Principle* and then, applying this theorem, show that the well known non-elementary integrals $\int \exp(x^2)$ and $\int \frac{1}{\log x}$ are indeed non-elementary.

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