Symbolic Integration in Computer Algebra

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Integration

Geometric/Analytic Definition

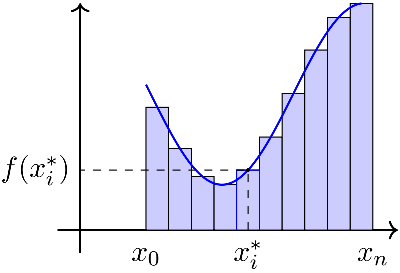
(i) $f : [a, b] \to \mathbb{R}$.

(ii) $(x_0, x_1, \ldots, x_n)$ a partition of $[a, b]$, with $a = x_0 < x_1 < \cdots < x_n = b$.

(iii) $\Delta x_i := x_i - x_{i-1}$ and $\Delta x := \max_{i=1}^{n} \Delta x_i$.

(iv) $x_i^* \in [x_{i-1}, x_i]$.

Integration



Definition (Integration)

$$\int_a^b f(x)\,dx := \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i.$$

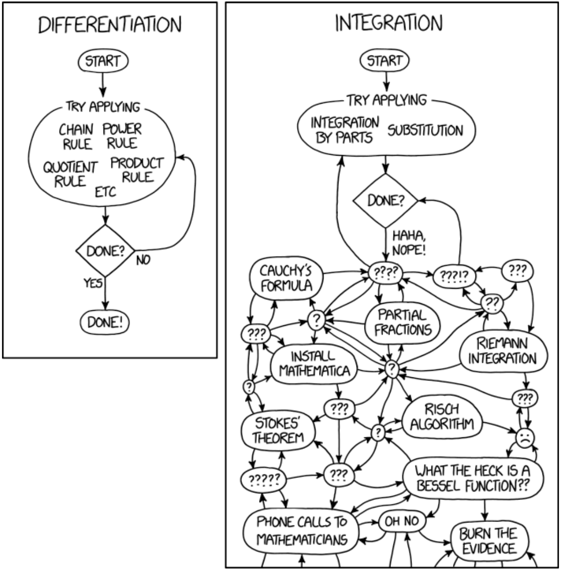
Whenever such a limit exists, we say f is integrable.

Integration

Theorem (Fundamental Theorem of Calculus)

Let $f : [a, b] \to \mathbb{R}$ be integrable and $F(x) := \int_a^x f(t)\,dt$. Then $F'(x) = f(x)$ and $\int_a^b f(x)\,dx = F(b) - F(a)$.

Integration



xkcd.com/2117/

Polynomials

Let F be a field.

Definition (Monic)

We say $f \in F[x]$ is *monic* if it is non-zero and the *leading coefficient* of f is 1.

Polynomials

Let F be a field.

Definition (GCD)

The *greatest common divisor* $\gcd(f, g)$ of $f, g \in F[x]$ is the (unique) polynomial, maximum with respect degree, amongst all *monic* polynomials dividing both f and g.

Polynomials

Let F be a field.

Definition (Square-free)

A nonzero polynomial $f \in F[x]$ is called *square-free* if there is no $q \in F[x] - F$ with $q^2 \mid f$.

Example

Let $f := x^3 - 2x^2 + x \in \mathbb{Q}[x]$. Notice $f = x(x-1)^2$, so $(x-1)^2 \mid f$.

Polynomials

Let F be a field.

Definition (Irreducible)

Let $f, g, h \in F[x]$. We say that f is irreducible if $f \notin F$ and $f = gh$ implies that $g \in F$ or $h \in F$.

Theorem (Existence and Uniqueness of Factorisations)

Let $f \in F[x] - \{0\}$. Then there exist unique $c \in F$, monic irreducibles $g_1, \ldots, g_m \in F[x]$, and positive integers $r_1, \ldots, r_m$ with

$$f = r \cdot g_1^{r_1} \cdots g_m^{r_m}.$$

Square-free Integration

Problem

Let $a, b \in \mathbb{Q}[x] - \{0\}$ with $\gcd(a, b) = 1$ and b monic and square-free.

$$\int \frac{a}{b}$$

Square-free Integration

Example

$$\int \frac{1}{x^2 - 3x + 2}$$

Solution

$$\frac{1}{x^2 - 3x + 2} = \frac{A}{x-1} + \frac{B}{x-2}$$

$$A = -B = -1$$

$$\begin{aligned}
\int \frac{1}{x^2 - 3x + 2} &= \int \frac{-1}{x-1} + \int \frac{1}{x-2} \\
&= -\log(x-1) + \log(x-2)
\end{aligned}$$

Square-free Integration

Example

$$\int \frac{1}{x^2 - 2}$$

Solution

Work in $\mathbb{Q}(\sqrt{2})$.

$$\frac{1}{x^2 - 2} = \frac{A}{x - \sqrt{2}} + \frac{B}{x + \sqrt{2}}$$

$$A = -B = \frac{\sqrt{2}}{4}$$

Square-free Integration

$$\begin{aligned}\int \frac{1}{x^2-2} &= \int \frac{\frac{\sqrt{2}}{4}}{x-\sqrt{2}} + \int \frac{-\frac{\sqrt{2}}{4}}{x+\sqrt{2}} \\ &= \frac{\sqrt{2}}{4}\log(x-\sqrt{2}) - \frac{\sqrt{2}}{4}\log(x+\sqrt{2})\end{aligned}$$

Square-free Integration

Theorem (Partial Fractions)

Let F be a field and $f, g \in F[x] - \{0\}$ with g monic and $\gcd(f,\, g) = 1$. Write g be the product of distinct irreducibles

$$g = \prod_{i=1}^{k} p_i^{n_i}.$$

There are unique $b, a_{ij} \in F[x]$ with $\deg a_{ij} < \deg p_i$ such that

$$\frac{f}{g} = b + \sum_{i=1}^{k} \sum_{j=1}^{n_i} \frac{a_{ij}}{p_i^j}$$

Square-free Integration

Theorem (Kronecker)

Let F be a field and $p \in F[x] - F$. There exists an extension G of F such that p has a root in G.

Proof sketch.

If p has no roots in F, let q be a monic irreducible dividing p. Then $K := F[x]/\langle q \rangle$ is a field, and $x + \langle q \rangle$ is a root of q (and therefore of p). □

Corollary

There exists an extension K of F such that p has $\deg p$ roots (counted with multiplicities) in K. We call such a field K a *splitting field* of p.

Square-free Integration

Let $g = (x - c_1) \cdots (x - c_n) \in K[x]$.

By the *partial fractions theorem*, there are $b, a_i \in K[x]$ with

$$\frac{f}{g} = b + \sum_{i=1}^{n} \frac{a_i}{x - c_i}.$$

Therefore

$$\int \frac{f}{g} = \int b + \sum_{i=1}^{n} a_i \log(x - c_i).$$

Splitting Fields

Theorem

Let K be a field and $p \in K[x]$ be monic and irreducible. The field $K[x]/\langle p \rangle$ has *finite dimension* $\deg p$ over K.

Proof sketch.

Consider the set $\{1, x, x^2, \ldots, x^k\}$. If $k \geq \deg p$, the coefficients of p give us a linear combination to 0. Otherwise, the existence of such a combination contradicts p being minimal. $\square$

Corollary

Let K be a field and $f \in K[x] - K$ have degree n. The dimension of the (smallest) splitting field of f over K may be as large as $n!$.

Example

Consider $x^3 - 2 \in \mathbb{Q}[x]$. The (smallest) splitting field is $\mathbb{Q}(\sqrt[3]{2}, e^{2\pi i/3})$, which has dimension $3!$ over $\mathbb{Q}$.

Splitting Fields

Example

$$\int \frac{4x^2+2x-4}{x^3+x^2-2x-2}$$

Solution

In $\mathbb{Q}$:

$$x^3+x^2-2x-2=(x+1)(x^2-2)$$

In $\mathbb{Q}(\sqrt{2})$:

$$(x+1)(x^2-2)=(x+1)(x+\sqrt{2})(x-\sqrt{2})$$

Splitting Fields

Partial fraction decomposition is:

$$\frac{4x^2 + 2x - 4}{x^3 + x^2 - 2x - 2} = \frac{2}{x+1} + \frac{1}{x - \sqrt{2}} + \frac{1}{x + \sqrt{2}}$$

Therefore:

$$\begin{aligned}\int \frac{4x^2 + 2x - 4}{x^3 + x^2 - 2x - 2} &= 2\log(x+1) + \log(x - \sqrt{2}) + \log(x + \sqrt{2}) \\ &= 2\log(x+1) + \log(x^2 - 2)\end{aligned}$$

Efficient Square-free Integration

Definition

Let $a, b \in F[x]$ with

$$a = \sum_{i=0}^{m} a_i x^i, \qquad a_m \neq 0$$

$$b = \sum_{i=0}^{n} b_i x^i, \qquad b_n \neq 0$$

The resultant of a and b may be defined

$$\operatorname{res}(a,\, b) = a_m^n b_n^m \prod_{\substack{1 \le i \le m \\ 1 \le j \le n}} (\lambda_i - \mu_j)$$

where λ_i are the roots of a and μ_j are the roots of b (counted with their multiplicities) over a splitting field of ab.

Efficient Square-free Integration

Theorem (Rothstein-Trager)

Let K be a finitely (and explicitly) generated algebraic extension of $\mathbb{Q}$ and $a, b \in K[x]$ with $\deg a < \deg b$, $\gcd(a,\, b) = 1$, b monic and square-free.

Let $R(z) = \mathrm{res}_x(a - zb',\, b) \in K[z]$. Let $\{c_i\}_{i=1}^m$ be the distinct roots of R over its minimal splitting field K^*. Then

$$\int \frac{a}{b} = \sum_{i=1}^{m} c_i \log(\gcd(a - c_i b',\, b))$$

Rothstein-Trager Method

Example

$$\int \frac{4x^2 + 2x - 4}{x^3 + x^2 - 2x - 2}$$

Solution

Let $a = 4x^2 + 2x - 4$ and $b = x^3 + x^2 - 2x - 2$.

$R(z) = \mathrm{res}_x(a - zb', b) = 8z^3 - 32z^2 + 40z - 16$.

Splitting field is $\mathbb{Q}$, with (distinct) roots $c_1 = 1$ and $c_2 = 2$.

$$\begin{aligned}\int \frac{a}{b} &= \log(\gcd(a - b', b)) + 2\log(\gcd(a - 2b', b)) \\ &= \log(x^2 - 2) + 2\log(x + 1)\end{aligned}$$

Thanks for listening!

Questions?