# Algorithmic Symbolic Integration

## An Introduction

Mitchell Holt

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# Integration

Geometric/Analytic Definition

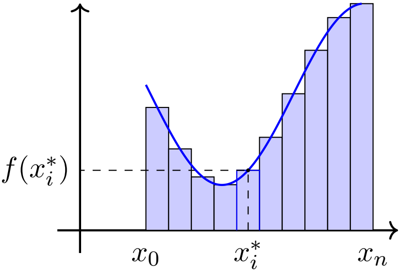
(i) $f : [a, b] \to \mathbb{R}$.

(ii) $P := (x_0, x_1, \ldots, x_n)$ a partition of $[a, b]$, with $a = x_0 < x_1 < \cdots < x_n = b$.

(iii) $\Delta x_i := x_i - x_{i-1}$ and $\Delta x := \max_{i=1}^{n} \Delta x_i$.

(iv) $x_i^* \in [x_{i-1}, x_i]$.

# Integration




### Definition (Integration)

$$\int_a^b f(x)\,dx := \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i.$$

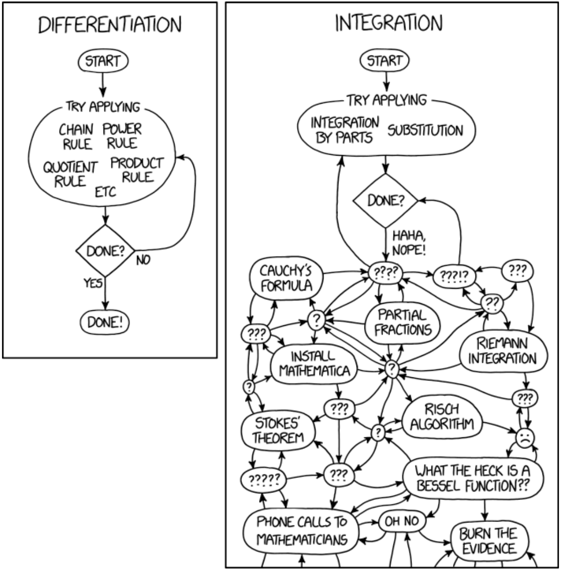
Whenever such a limit exists, we say $f$ is integrable.

# Integration

**Theorem (Fundamental Theorem of Calculus)**

Let $f : [a, b] \to \mathbb{R}$ be integrable and $F(x) := \int_a^x f(t)\, dt$. Then $F'(x) = f(x)$ and $\int_a^b f(x)\, dx = F(b) - F(a)$.

# Integration




xkcd.com/2117/

# Integration in Magma

Example

$$\int \frac{6x^5 - 4x^4 - 32x^3 + 12x^2 + 34x - 24}{x^6 - 8x^4 + 17x^2 - 8}\, dx$$

# Integration in Magma

```
> itg := IntegrateAssignNames(f/g);
> itg;
a*log(x^3 + (a - 1)*x^2 - 3*x - 2*a + 2) +
    (-a + 2)*log( x^3 + (-a + 1)*x^2 - 3*x
    + 2*a - 2)
```

$$\int \frac{6x^5 - 4x^4 - 32x^3 + 12x^2 + 34x - 24}{x^6 - 8x^4 + 17x^2 - 8}\, dx$$

$$= a\log(x^3 + (a-1)x^2 - 3x - 2a + 2)$$

$$+ (2-a)\log(x^3 + (1-a)x^2 - 3x + 2a - 2)$$

```
> DefiningPolynomial(ConstantField(Parent(
   itg)));
y^2 - 2*y - 1
```

# Easy Wins

(1) Polynomials from $\mathbb{Q}[x]$.

(2) Lookup table: $\sin x$, $\cos x$, $e^x$, $\ldots$

(3) $\displaystyle\int \frac{u'}{u} = \log u$, $u \in \mathbb{Q}[x]$.

# Integration Techniques

## Problem

$$\int \frac{1}{x^2 - 3x + 2}$$

## Solution

$$\frac{1}{x^2 - 3x + 2} = \frac{A}{x - 1} + \frac{B}{x - 2}$$

$$A = -B = -1$$

$$\begin{aligned} \int \frac{1}{x^2 - 3x + 2} &= \int \frac{-1}{x - 1} + \int \frac{1}{x - 2} \\ &= -\log(x - 1) + \log(x - 2) \end{aligned}$$

# Partial Fractions

## Problem

$$\int \frac{1}{x^2 - 2}$$

## Solution

Work in $\mathbb{Q}(\sqrt{2})$.

$$\frac{1}{x^2 - 2} = \frac{A}{x - \sqrt{2}} + \frac{B}{x + \sqrt{2}}$$

$$A = -B = \frac{\sqrt{2}}{4}$$

# Partial Fractions

$$\begin{aligned}\int \frac{1}{x^2-2} &= \int \frac{\frac{\sqrt{2}}{4}}{x-\sqrt{2}} + \int \frac{-\frac{\sqrt{2}}{4}}{x+\sqrt{2}} \\ &= \tfrac{\sqrt{2}}{4}\log(x-\sqrt{2}) - \tfrac{\sqrt{2}}{4}\log(x+\sqrt{2})\end{aligned}$$

# Partial Fractions

### Theorem (Partial Fractions)

Let $f, g \in \mathbb{Q}[x] - \{0\}$. Write $g$ be the product of distinct irreducibles

$$g = \prod_{i=1}^{k} p_i^{n_i}.$$

There are unique $b, a_{ij}$ with $\deg a_{ij} < \deg p_i$ such that

$$\frac{f}{g} = b + \sum_{i=1}^{k} \sum_{j=1}^{n_i} \frac{a_{ij}}{p_i^j}$$

# Easy Wins

(1) Polynomials from $\mathbb{Q}[x]$.

(2) Lookup table: $\sin x$, $\cos x$, $e^x$, $\ldots$

(3) $\displaystyle\int \frac{u'}{u} = \log u$, $u \in \mathbb{Q}[x]$.

(4) *Partial fractions*.

# Partial Fractions

## Problem

$$\int \frac{1}{(x-1)^2}$$

Already in partial fraction expansion form!

$$\frac{1}{(x-1)^2} \neq \frac{u'}{u}$$

# Square-free Denominators

Let $f, g \in \mathbb{Q}[x] - \{0\}$ with $g$ square-free and $\gcd(f,\, g) = 1$.

## Problem

$$\int \frac{f}{g}$$

## Theorem (Kronecker)

Let $F$ be a field and $p \in F[x]$ with $\deg p > 0$. There exists an extension $G$ of $F$ such that $p$ has a root in $G$.

## Corollary

There exists an extension $K$ of $F$ such that $p$ has $\deg p$ roots in $K$. We say such a field $K$ a *splitting field* of $p$.

# Square-free Denominators

Let $g = (x - c_1) \cdots (x - c_n) \in K[x]$.

By the *partial fractions* theorem, there are $b, a_i \in K[x]$ with:

$$\frac{f}{g} = b + \sum_{i=1}^{n} \frac{a_i}{x - c_i}$$

$$\int \frac{f}{g} = \int b + \sum_{i=1}^{n} a_i \log(x - c_i)$$

# Easy Wins

(1) Polynomials from $\mathbb{Q}[x]$.

(2) Lookup table: $\sin x$, $\cos x$, $e^x$, $\ldots$

(3) $\displaystyle\int \frac{u'}{u} = \log u$, $u \in \mathbb{Q}[x]$.

(4) Partial fractions *when the denominator is square-free*.

# Hermite's Method

## Problem

$$\int \frac{4x^7 - 16x^6 + 28x^5 - 351x^3 + 588x^2 - 738}{2x^7 - 8x^6 + 14x^5 - 40x^4 + 82x^3 - 76x^2 + 120x - 144}$$

# Hermite's Method

### Theorem (Square-free Decomposition)

Let $F$ be a field and $p \in F[x] - \{0\}$. There exist unique square-free monic polynomials $\{q_i\}_{i=1}^m$ in $F[x]$ and $r \in F$ such that $\gcd(q_i, q_j) = 1$ whenever $i \neq j$ and

$$p = r \prod_{i=1}^{m} q_i^i.$$

We call this the *square-free decomposition* of $p$.

# Hermite's Method

$$\int \frac{f}{g} = \int \frac{4x^7 - 16x^6 + 28x^5 - 351x^3 + 588x^2 - 738}{2x^7 - 8x^6 + 14x^5 - 40x^4 + 82x^3 - 76x^2 + 120x - 144}$$

```
> P<x> := PolynomialRing(RationalField());
> f := 4*x^7 - 16*x^6 + 28*x^5 - 351*x^3 +
    588*x^2 - 738;
> g := 2*x^7 - 8*x^6 + 14*x^5 - 40*x^4 +
   82*x^3 - 76*x^2 + 120*x - 144;
> GCD(f, g);
1
> SquarefreeFactorisation(g);
[ <x^2 + x + 3, 2>, <x - 2, 3> ]
```

$$g = (x^2 + x + 3)^2 (x - 2)^3$$

# Hermite's Method

**Theorem (Square-free Partial Fraction Decomposition)**

Suppose $F$ is a field and $f, g \in F[x] - \{0\}$ with $\gcd(f, g) = 1$. Without loss of generality, assume that $g$ is monic such that $g = \prod_{i=1}^{m} q_i^i$ the square-free decomposition of $g$. Then there exist polynomials $b$ and $r_{ij}$, $1 \leq j \leq i$, $1 \leq i \leq m$ such that

$$\frac{f}{g} = b + \sum_{i=1}^{m} \sum_{j=1}^{i} \frac{r_{ij}}{q_i^j}$$

and $\deg r_i < \deg q_i$ for all $i = 1, \ldots, m$.

# Hermite's Method

```
> sfpf := SquarefreePartialFractionDecomposition(f/g);
> sfpf;
[
     <1, 1, 2>,
     <x^2 + x + 3, 2, 45*x + 45/2>,
     <x - 2, 3, -5>
]
> f/g eq &+[ tm[3]/(tm[1]^tm[2]) : tm in sfpf];
true
```

$$\frac{f}{g} = 2 + \frac{45x + \frac{45}{2}}{(x^2 + x + 3)^2} + \frac{-5}{(x-2)^3}$$

# Hermite Reduction

## Problem

$$\int \frac{r}{q^j}, \qquad q \text{ square-free.}$$

Base case: $j = 1$.

## Theorem

Let $F$ be a field and $p \in F[x]$. Then $p$ is square-free if and only if $\gcd(p, \frac{d}{dx}p) = 1$.

## Theorem (Bèzout's Lemma)

Let $F$ be a field and $p, q, g \in F[x]$. There exist polynomials $a, b \in F[x]$ such that $ap + bq = g$ if and only if $\gcd(p, q) \mid g$. Moreover, if any $a, b$ exist, there are particular solutions with $\deg a < \deg q$ and $\deg b < \deg p$.

# Hermite Reduction

Let $s, t \in F[x]$ satisfy $sq + tq' = r$.

$$\int \frac{r}{q^j} = \int \frac{s}{q^{j-1}} + \int \frac{tq'}{q^j}$$

# Hermite Reduction

## Problem

$$\int \frac{tq'}{q^j}$$

*Integration by parts*: $\int uv' = uv - \int u'v$.

$$u = t, \qquad v = -\frac{1}{(j-1)q^{j-1}}$$

$$\int \frac{tq'}{q^j} = \frac{-t/(j-1)}{q^{j-1}} + \int \frac{t'/(j-1)}{q^{j-1}}$$

# Hermite Reduction

Therefore,

$$\begin{aligned}
\int \frac{r}{q^j} &= \int \frac{s}{q^{j-1}} + \int \frac{tq'}{q^j} \\
&= \int \frac{s}{q^{j-1}} + \frac{-t/(j-1)}{q^{j-1}} + \int \frac{t'/(j-1)}{q^{j-1}} \\
&= \frac{-t/(j-1)}{q^{j-1}} + \int \frac{s + t'/(j-1)}{q^{j-1}}
\end{aligned}$$

# Hermite's Method

## Problem

$$\int \frac{f}{g} = \int \frac{4x^7 - 16x^6 + 28x^5 - 351x^3 + 588x^2 - 738}{2x^7 - 8x^6 + 14x^5 - 40x^4 + 82x^3 - 76x^2 + 120x - 144}$$

$$\frac{f}{g} = 2 + \frac{45x + \frac{45}{2}}{(x^2 + x + 3)^2} + \frac{-5}{(x-2)^3}$$

# Hermite's Method

## Problem

$$\int \frac{-5}{(x-2)^3}$$

```
> r := -5;
> q := x - 2;
> j := 3;
> _, s, t := XGCD(q, Derivative(q));
> s *:= r; t *:= r;
> -t/(j - 1), s + Derivative(t)/(j - 1);
5/2
0
```

$$\frac{\frac{5}{2}}{(x-2)^2} + \int 0$$

# Hermite's Method

## Problem

$$\int \frac{45x + \frac{45}{2}}{(x^2 + x + 3)^2}$$

```
> r := 45*x + 45/2;
> q := x^2 + x + 3;
> j := 2;
> _, s, t := XGCD(q, Derivative(q));
> s *:= r; t *:= r;
> -t/(j - 1), s + Derivative(t)/(j - 1);
90/11*x^2 + 90/11*x + 45/22
0
```

$$\frac{\frac{90}{11}x^2 + \frac{90}{11}x + \frac{45}{22}}{x^2 + x + 3} + \int 0$$

# Hermite's Method

$$\int \frac{4x^7 - 16x^6 + 28x^5 - 351x^3 + 588x^2 - 738}{2x^7 - 8x^6 + 14x^5 - 40x^4 + 82x^3 - 76x^2 + 120x - 144}$$
$$= \int 2 + \int \frac{45x + \frac{45}{2}}{(x^2 + x + 3)^2} + \int \frac{-5}{(x-2)^3}$$
$$= 2x + \frac{\frac{5}{2}}{(x-2)^2} + \frac{\frac{90}{11}x^2 + \frac{90}{11}x + \frac{45}{22}}{x^2 + x + 3}$$

# Easy Wins

(1) Polynomials from $\mathbb{Q}[x]$.

(2) Lookup table: $\sin x$, $\cos x$, $e^x$, $\ldots$

(3) $\displaystyle\int \frac{u'}{u} = \log u$, $u \in \mathbb{Q}[x]$.

(4) Partial fractions *when the denominator is square-free.*

(5) *Hermite reduction*.

# Efficient Square-free Integration

## Definition

Let $a, b \in F[x]$ with

$$a = \sum_{i=0}^{m} a_i x^i, \qquad a_m \neq 0$$

$$b = \sum_{i=0}^{n} b_i x^i, \qquad b_n \neq 0$$

The resultant of $a$ and $b$ may be defined

$$\operatorname{res}(a,\, b) = a_m^n b_n^m \prod_{\substack{1 \le i \le m \\ 1 \le j \le n}} (\lambda_i - \mu_j)$$

where $\lambda_i$ are the roots of $a$ and $\mu_j$ are the roots of $b$ (counted with their multiplicities) over a splitting field of $ab$.

# Efficient Square-free Integration

## Theorem (Rothstein-Trager)

Let $K$ be a finitely (and explicitly) generated algebraic extension of $\mathbb{Q}$ and $a, b \in K[x]$ with $\deg a < \deg b$, $\gcd(a,\, b) = 1$, $b$ monic and square-free.

Let $R(z) = \operatorname{res}_x(a - zb',\, b) \in K[z]$. Let $\{c_i\}_{i=1}^{m}$ be the distinct roots of $R$ over its minimal splitting field $K^*$. Then

$$\int \frac{a}{b} = \sum_{i=1}^{m} c_i \log(\gcd(a - c_i b',\, b))$$

# Rothstein-Trager

## Problem

$$\int \frac{4x^2 + 2x - 4}{x^3 + x^2 - 2x - 2}$$

## Solution

```
> a := 4*x^2 + 2*x - 4;
> b := x^3 + x^2 - 2*x - 2;
> PP<y, z> := PolynomialRing(RationalField(), 2);
> h := hom< P -> PP | y >;
> r := Resultant(h(a) - z*h(Derivative(b)), h(b), 1);
> r;
8*z^3 - 32*z^2 + 40*z - 16
```

# Rothstein-Trager

```
> rts := Roots(UnivariatePolynomial(r));
> rts
[ <1, 2>, <2, 1> ]
> [<rt[1], GCD(a - rt[1]*Derivative(b), b)
   > : rt in rts];
[
    <1, x^2 - 2>,
    <2, x + 1>
]
```

$$\int \frac{4x^2 + 2x - 4}{x^3 + x^2 - 2x - 2} = \log(x^2 - 2) + 2\log(x + 1)$$

Thanks for listening!

Questions?

# Elementary Functions

## Definition (Elementary Function Field)

Let $K = \mathbb{Q}(\alpha_1, \dots, \alpha_m)$ be a constant differential field (with differential operator $'$) where each $\alpha_i$ is algebraic over $\mathbb{Q}$. Let $x$ be a transcendental symbol over $K$ satisfying $x' = 1$ (in the differential field $(K(x), ')$). In the differential extension field $K(x, \theta_1, \dots, \theta_n)$, we say each $\theta_j$ $(j = 1, \dots, n)$ is *elementary* over $K(x, \theta_1, \dots, \theta_{j-1})$ if $\theta_j$ is *transcendental* over $K(x, \theta_1, \dots, \theta_{j-1})$ and either:

(i) $\theta_j' = u'/u$ for some $u \in K(x, \theta_1, \dots, \theta_{j-1})$ (in which case we say $\theta_j$ is *logarithmic*), or

(ii) $\theta_j'/\theta_j = u'$ for some $u \in K(x, \theta_1, \dots, \theta_{j-1})$ (in which case we say $\theta_j$ is *exponential*).

If each $\theta_j$ is elementary, then we say that $K(x, \theta_1, \dots, \theta_{j-1})$ is an *elementary function field.*

# Elementary Functions

### Definition (Elementary Extension)

Let $F$ be an elementary function field. We say $G$ is an *elementary extension* of $F$ if $G$ is an elementary function field, $G$ is a differential extension of $F$, and $G$ has the same constant field as $F$.

### Definition (Integration)

Let $F$ be an elementary function field and $f \in F$. If there exists a finitely and explicitly generated elementary extension $G$ of $F$ and a $g \in G$ with $g' = f$, then we say $g$ is the *elementary anti-derivative* or *elementary integral* of $f$ and write $\int f = g$. If no such $g$ or $G$ exist, we say that $f$ has no elementary integral.

# Liouville's Principle

**Theorem**

Let $F$ be an elementary function field and $\theta$ be transcendental and logarithmic over $F$, with $\theta' = \frac{u'}{u}$ for some $u \in F$. If $f \in F[\theta]$ with $\deg f > 0$, then:

(i) $f' \in F[\theta]$.

(ii) If the leading coefficient of $f$ is constant, then $\deg f' = \deg f - 1$.

(iii) If the leading coefficient of $f$ is *not* constant, then $\deg f' = \deg f$.

# Liouville's Principle

**Theorem**

Let $F$ be an elementary function field and $\theta$ be transcendental and exponential over $F$, with $\theta'/\theta = u'$ for some $u \in F$.

(i) For all $g \in F - \{0\}$ and $n \in \mathbb{Z} - \{0\}$, there exists a $h \in F - \{0\}$ such that $(g\theta^n)' = h\theta^n$.

(ii) If $f \in F[\theta]$ with $\deg f > 0$, then $f' \in F[\theta]$ and $\deg f' = \deg f$.

# Liouville's Principle

### Theorem (Liouville's Principle)

Let $(F, D)$ be a differential field and $f \in F$. If there exists an elementary extension $(G, E)$ of $F$ and a $g \in G$ such that $Eg = f$, then there exist $v_0, \ldots, v_m \in F$ and $c_1, \ldots, c_m$ in the constant field of $G$ such that

$$f = Dv_0 + \sum_{i=1}^{m} c_i \frac{Dv_i}{v_i}.$$

### Proof sketch.

Let $G = F(\theta_1, \ldots, \theta_n)$ and proceed by induction on $n$. Base case is $n = 0$. For the inductive step, let $g = \frac{a}{b}$ with $a, b \in K(x, \theta_1, \ldots, \theta_k)[\theta_{k+1}]$ and apply previous theorems. $\square$

# Logarithmic Integration

## Problem

Let $f, g \in F[\theta]$ where $\theta$ elementary and logarithmic over $F$.

$$\int \frac{f}{g}$$

## Solution

Use a variation of *Hermite's Method* to find $h \in F(\theta)$ and $p, a, b \in F[\theta]$ with $\deg a < \deg b$ and $b$ monic and square-free, and

$$\int \frac{f}{g} = h + \int p + \int \frac{a}{b}.$$

# Logarithmic Integration

## Theorem (Rothstein-Trager — Logarithmic Case)

Let $F$ be an elementary function field and with constant field $K$. Let $\theta$ be elementary and logarithmic over $F$ and suppose that $F(\theta)$ has the same constant field $K$. Let $\frac{a}{b} \in F(\theta)$ where $a, b \in F[\theta]$ with $\gcd(a, b) = 1$ and $b$ monic and square-free. Let $R = \operatorname{res}_\theta(a - zb', b) \in F[z]$.

(i) $\int \frac{a}{b}$ is elementary if and only if all of the roots of $R$ are constants (equivalently, $R/\operatorname{lc}(R) \in K[z]$).

(ii) If $\int \frac{a}{b}$ is elementary then

$$\frac{a}{b} = \sum_{i=1}^{m} c_i \frac{v_i'}{v_i}, \tag{1}$$

where $\{c_i\}_{i=1}^{m}$ are the distinct roots of $R$ over its splitting field and $v_i = \gcd(a - c_i b', b)$ for each $i = 1, \ldots, m$.

# Logarithmic Integration

Let $p = \sum_{i=0}^{m} p_i \theta^i \in F[\theta]$.

$$\int p = \sum_{i=0}^{m+1} q_i \theta^i + \sum_{j=1}^{n} c_j \log v_j,$$

$q_{m+1} \in K$, $q_0, \ldots, q_m \in F$, $v_j \in F$, $c_j \in K$.

$$\sum_{i=0}^{m} p_i \theta^i = q_{m+1}\theta^{m+1} + \sum_{i=1}^{m} (q_i' + (i+1) q_{i+1}' \theta') \theta^i + q_1 \theta' + q_0 + \sum_{j=1}^{n} c_j \frac{v_j'}{v_j},$$

# Logarithmic Integration

## Problem

Let $F = \mathbb{Q}(x)$ and $\theta = \log x$.

$$\int (2\theta + 2)$$

## Solution

$$\int (2\theta + 2) = q_2\theta^2 + q_1\theta + \bar{q}_0,$$

$q_2, q_1, \bar{q}_0 \in \mathbb{Q}(x)$, $q_2$ constant.

$$2\theta + 2 = (q_1' + 2q_2\theta')\theta + q_1' + \bar{q}_0.$$

# Logarithic Integration

$$
\begin{aligned}
2 &= q_1' + 2q_2\theta' \\
\implies q_1 &= \int 2 - 2q_2 \int \theta' = 2x - 2q_2\theta \\
\implies q_2 &= 0 \quad \text{and} \quad q_1 = 2x
\end{aligned}
$$

$$
\begin{aligned}
2 &= q_1' + \bar{q}_0 = 2 + \bar{q}_0 \\
\implies \bar{q}_0 &= 0
\end{aligned}
$$

Therefore

$$\int (2\theta + 2) = 2x\theta \in F(\theta).$$

# Exponential Integration

## Problem

Let $F$ be an elementary function field and $\theta$ elementary and exponential over $F$.

$$\int \sum_{i=-k}^{m} p_i \theta^i$$

## Solution

$$\int \sum_{i=-k}^{m} p_i \theta^i = \sum_{i=-k}^{m} q_i \theta^i + \sum_{j=1}^{n} c_j \log v_j$$

$$\implies \sum_{i=-k}^{m} p_i \theta^i = \sum_{i=-k}^{m} (q_i' + i q_i \theta') \theta^i + \sum_{j=1}^{n} c_j \frac{v_j'}{v_j}$$

# Exponential Integration

$$p_i = q_i' + i\theta' q_i$$

*Risch Differential Equation*

$$y' + fy = g, \qquad f, g \in F$$