AMSI SUMMER RESEARCH SCHOLARSHIPS 2024–25

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Modular Algorithms in Quadratic Algebraic Number Fields

Mitchell Holt

Supervised by Dr Paul Vrbik

The University of Queensland

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Computer Algebra

Theorem (Chinese Remainder Theorem (CRT))

Essentially... there is a u satisfying

$$\begin{aligned} u &\equiv u_1 \pmod{m_1}, \\ &\vdots \\ u &\equiv u_n \pmod{m_n}, \end{aligned}$$

when m_i and m_j are coprime.

Proof sketch.

$$\begin{aligned}\phi : \mathbb{Z}_{m_1 \cdots m_n} &\to \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_n} \\ u &\mapsto (u \mod m_1, \ldots, u \mod m_n)\end{aligned}$$

is an isomorphism. $\square$

Find u with

$$
\begin{aligned}
u &\equiv 49 \pmod{32771},\\
u &\equiv -21 \pmod{65537},\\
u &\equiv -30 \pmod{131101}.
\end{aligned}
$$

$$u = v_0 + v_1(m_1) + v_2(m_1 m_2) + \cdots + v_{n-1}\Big(\prod_{i=1}^{n-1} m_i\Big)$$

$$-104905043721354 = 49 - 28(32771) - 48845(32771 \cdot 65537)$$

Modular Algorithms

$$
\begin{aligned}
22x + 44y + 74z &= 1,\\
15x + 14y - 10z &= -2,\\
-25x - 28y + 20z &= 34.
\end{aligned}
$$

Fraction-free Gaussian elimination:

$$
\begin{aligned}
1257315840x &= 7543895040, \\
-57150720y &= 314328960, \\
162360z &= 243540.
\end{aligned}
$$

$$
\left\{ x = 6, \qquad y = -\frac{11}{2}, \qquad z = \frac{3}{2} \right\}
$$

Solve $A\vec{x} = \vec{b}$ [TI61],[CL77].

$p_1, \ldots, p_k \leftarrow$ distinct primes

for $i = 1, \ldots, n$ **do**

  $d_i \leftarrow \det A \pmod{p_i}$

  $\vec{x}_i \leftarrow$ solve $A\vec{x} = \vec{b} \pmod{p_i}$

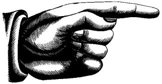
$D \leftarrow$ CRT on $\{d_1 \pmod{p_1}, \ \ldots, \ d_n \pmod{p_n}\}$

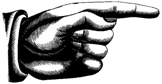
$\vec{y} \leftarrow$ pointwise CRT on $\{d_1 \cdot \vec{x}_1 \pmod{p_1}, \ \ldots, \ d_n \cdot \vec{x}_n \pmod{p_n}\}$

return $\frac{1}{D} \cdot \vec{y}$

Modular linear solver:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{-7380}\begin{bmatrix} -44280 \\ 40590 \\ -11070 \end{bmatrix} = \begin{bmatrix} 6 \\ -11/2 \\ 3/2 \end{bmatrix}$$

$\mathbb{Q}$  $\mathbb{Z}$

$\mathbb{Q}(\sqrt{M})$  ?

Quadratic Integers

Let $M \in \mathbb{Z}$ be *squarefree*. The *quadratic integers* of $\mathbb{Q}(\sqrt{M})$ are

$$R = \begin{cases} \mathbb{Z}[\sqrt{M}] & M \equiv 2, 3 \pmod 4 \\ \mathbb{Z}[\frac{1+\sqrt{M}}{2}] & M \equiv 1 \pmod 4 \end{cases}$$

[Mar77]. Write $R = \mathbb{Z}[\gamma]$.

- What are the primes?
- Are any primes $p \in \mathbb{Z}$ also prime in R (*inert*)?

Chebotarëv Density Theorem implies infinitely many inert primes.

Algorithm: ([IR13] Proposition 5.2.2)

- Input: Set of odd primes P.
- Output: An odd inert prime not contained in P.

Proposition

Let $p \in \mathbb{Z}$ be an inert prime in $R = \mathbb{Z}[\gamma]$. Then

$$R/pR \simeq \begin{cases} \mathbb{Z}_p[x]/\langle x^2 - M \rangle & M \equiv 2, 3 \pmod 4 \\ \mathbb{Z}_p[x]/\langle x^2 - x - \frac{M-1}{4} \rangle & M \equiv 1 \pmod 4 \end{cases}$$

Moreover, the isomorphism is given by

$$\phi([a + b\gamma]) = [a + bx].$$

Modular Algorithms in $\mathbb{Q}(\sqrt{M})$

- Use Garner's integer CRT algorithm in $\mathbb{Z}[\gamma]$ (for inert primes).
- Use the same modular linear solver!

- No unique factorisation in R.
- No GCD.

- Rational reconstruction $\mathbb{Z}_k[\gamma] \rightsquigarrow \mathbb{Q}[\gamma] = \mathbb{Q}(\sqrt{M})$.

Solve $Ax = b$, where:

$$A = \begin{pmatrix} 81 - 19\sqrt{122} & 78 - 89\sqrt{122} & -81 - 80\sqrt{122} \\ 22 - 53\sqrt{122} & -8 + 66\sqrt{122} & -43 - 19\sqrt{122} \\ 50 - 30\sqrt{122} & -90 + 154\sqrt{122} & -2 - 124\sqrt{122} \end{pmatrix}, \quad b = \begin{pmatrix} 26851 - 2700\sqrt{122} \\ -41098 + 883\sqrt{122} \\ -67029 + 1076\sqrt{122}. \end{pmatrix}$$

$$x = \begin{pmatrix} \frac{5\sqrt{122}}{2} \\ \frac{33}{2} - 3\sqrt{122} \\ 15 \end{pmatrix}$$

- Fraction-free Gaussian elimination: 105-bit integers
- Modular solver: 57-bit integers (seven 16-bit primes)

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